Microstates at the boundary of AdS

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Abstract

The bound states of the D1D5 brane system have a known gravitational description: flat asymptotics, an anti-de Sitter region, and a 'cap' ending the AdS region. We construct perturbations that correspond to the action of chiral algebra generators on Ramond ground states of D1D5 branes. Abstract arguments in the literature suggest that the perturbation should be pure gauge in the AdS region; our perturbation indeed has this structure, with the nontrivial deformation of the geometry occurring at the 'neck' between the AdS region and asymptotic infinity. This 'non-gauge' deformation is needed to provide the nonzero energy and momentum carried by the perturbation. We also suggest implications this structure may have for the majority of microstates which live at the cap.

1 Introduction

Black holes have an entropy proportional to their surface area [1], suggesting that the states corresponding to this entropy should be localized around the horizon [2]. But in ordinary gravity theories we typically find no structure at the horizon – black holes 'have no hair' [3], a fact which leads to the information paradox [4], see also [5].

For black holes with a near-horizon AdS region, gauge/gravity duality [6, 7, 8] allows one to replace the entire AdS region by a CFT placed at the boundary of the AdS. In AdS_5 one encounters a separation between bulk degrees of freedom and a $U(1)$ degree of freedom localized at the boundary of AdS [8]. So one may ask: in the gravity description, which states (if any) are localized at the boundary of AdS, at the horizon, or deep inside the black hole?

We address this question in AdS_3, where it has been argued that the boundary diffeomorphisms of AdS_3 form a Virasoro algebra with a central charge c [9]. It has been further argued that this c should give the number of states in the 2D boundary theory at energy level N by the standard expression $\exp[2\pi\sqrt{\frac{cN}{6}}]$ [10]. Carlip has extended these ideas to general black hole horizons, and proposed ideas relating conformal symmetry to thermodynamic properties of black holes [11].

We study the D1D5 bound state of string theory, which generates a near-horizon AdS_3 geometry and yields a black hole in AdS_3 [12] when excitation energy is added. Suppose we start with an extremal state $|\psi\rangle$ of the D1D5 CFT which admits a classical gravitational description. Such geometries have the following schematic structure:
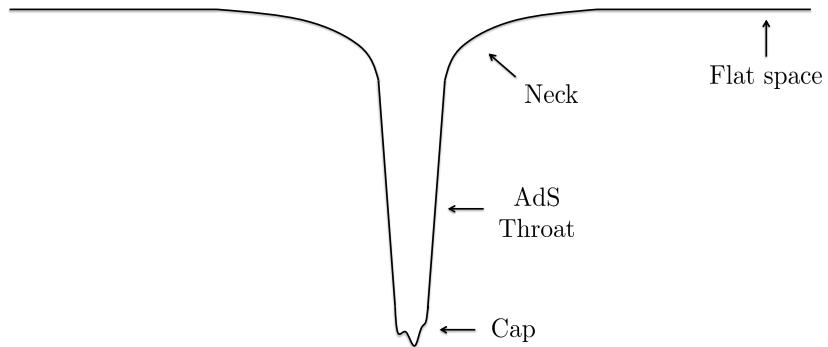

Figure 1: Structure of D1D5 geometries and associated terminology

There is flat space at infinity, then a 'neck' region which leads to a AdS_3 'throat'. The throat ends in a 'cap' which depends on the state $|\psi\rangle$ [13, 14, 15]. This structure has also been seen for non-extremal states [16]; for recent related work see [17, 18, 19].

In this paper we construct the perturbation corresponding to the state

$$L_{-n}|\psi\rangle$$

where L_{-n} is a Virasoro generator. The perturbed solution has D1, D5 and P charges. We find that to leading order the perturbation is a pure diffeomorphism in the AdS throat, but there is a genuine deformation at the neck region, which forms a natural boundary to the AdS throat. The perturbation:

(i) reduces to being a pure diffeomorphism in the AdS throat+cap;

(ii) is *not* pure gauge in the neck;

(iii) is normalizable at infinity;

(iv) is regular in the 'cap' region;

(v) has the correct spacetime dependence to carry the energy and momentum required by the excitation L_{-n}.

Thus our construction is consistent with the notion that in AdS_3 the action of the Virasoro algebra is represented by 'boundary diffeomorphisms' [9]. Note that the perturbation could not have been a pure diffeomorphism everywhere, since the state $L_{-n}|\psi\rangle$ has more energy than the state $|\psi\rangle$.
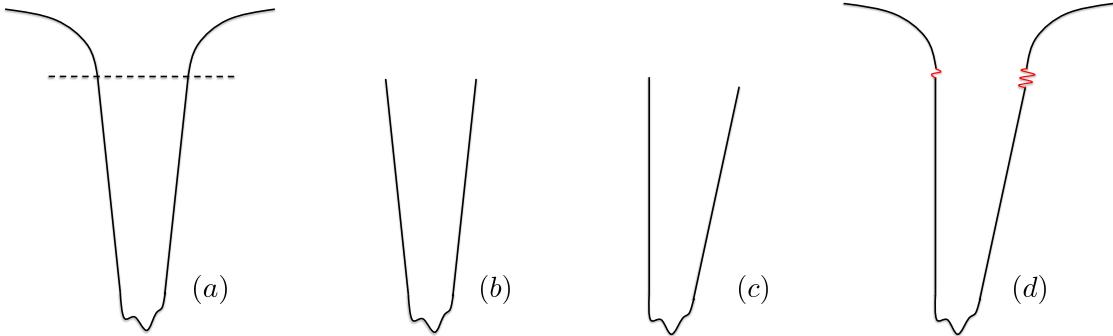

Figure 2: Cutting the AdS part of the geometry, performing a diffeomorphism and gluing it back, creating a real (i.e. not pure gauge) perturbation in the neck.

A cartoon of how one may imagine the perturbation is given in Fig. 2. Part (a) depicts the entire geometry, with a dashed line indicating the location where the boundary CFT will be placed. In (b) we truncate to the asymptotically AdS region that is dual to the CFT. In (c) we perform a diffeomorphism of region (b), corresponding to the application of a Virasoro generator. In (d) we glue this region back to flat space, creating a deformation in the 'neck' region.

In this paper we work in type IIB string theory compactified on $S^1 \times T^4$. We wrap n_1 D1 branes on S^1 and n_5 D5 branes on $S^1 \times T^4$, and so the full throat geometry is $AdS_3 \times S^3 \times T^4$. In addition to constructing the L_{-n} perturbation, we also carry out similar constructions for the left-moving $SU(2)$ current algebra generator $J^{(3)}_{-n}$ of the S^3 and the $U(1)$ currents J^i_{-n} which arise from the translations along the T^4.

We find our perturbation by solving the equations of motion separately in the outer region (throat+neck+flat infinity) and inner region (throat+cap), and then matching them to leading order in the throat. Such a matching procedure was carried out to several orders for a different state in [20].

For the case of the $U(1)$ currents J^i_{-n}, the solutions in the outer region can be seen to be a limit of the solutions in [21], where waves were added to a black string geometry. These latter solutions turned out to be singular at the horizon of the black string [22]. Such a singularity was encountered again in the construction of 'external hair' on extremal holes [23, 24], and it appears reasonable to attribute such singularities to the general 'no-hair theorem' which forbids smooth perturbations to a horizon. In the present work our geometry has a regular cap in place of a horizon, and so our perturbations do not become singular anywhere. We shall comment further on this issue in the discussion section.

The number of states generated by the chiral algebra is small, corresponding to a $U(1)$ degree of freedom with central charge $c = 6$. The remaining degrees of freedom, corresponding to a central charge $c = 6(n_1 n_5 - 1)$, are expected to live at the cap. Our construction of states at the neck suggests a construction of states localized in the cap, which we discuss in Section 7.

This paper is organized as follows:

- In Section 2 we review the 2-charge background solution, the equations to be solved, and describe the perturbative matching process.

- In Section 3 we derive the perturbations for the L_{-n} generators by solving the equations of motion in the neck and cap, and show that they agree in their domain of overlap.

- Sections 4 and 5 do the same for perturbations of the S^3 and T^4 respectively.

- Section 6 gives a cross-check of our results against a test of AdS/CFT.

- Section 7 discusses our results.

- Some technical details are placed in the appendices.

2 The 2-charge solution and the perturbation

2.1 The 2-charge solution

We take type IIB string theory with the compactification

$$M_{9,1} \; \rightarrow \; M_{4,1} \times S^1 \times T^4 \,. \tag{2.1}$$

We wrap n_1 D1 branes on S^1 and n_5 D5 branes on $S^1 \times T^4$. This D1D5 system can be mapped by dualities to the NS1P system, where the D5 branes become a multiwound fundamental string along the S^1 and the D1 branes become momentum P carried as travelling waves on the string. The gravitational solution of such a string carrying waves is known [25, 13], and may be dualized back to obtain gravitational solutions describing the D1D5 bound states [13, 26]. We use light-cone coordinates

$$v = t - y \,, \qquad u = t + y \tag{2.2}$$

constructed from the time t and S^1 coordinate y. The D1D5 solutions are given in terms of a profile function $\vec{F}(v)$ describing the vibration profile of the NS1 in the NS1P duality frame. Here $\vec{F}$ is the transverse displacement of the vibrating string and the dependence on v tells us that the waves move only in one direction along the string, generating a BPS state.

The generic state in this system is quantum in nature and the gravitational field it sources is not well-described by a classical geometry. This is simply because the momentum on the NS1 is spread over many different harmonics, with an occupation number of order unity per harmonic. But we can gain insight into the physics of generic solutions by starting with special ones, where all the momentum is placed in a few harmonics. In this case we can take coherent states for the vibrating NS1 string, which are now described by a classical profile function $\vec{F}(v)$ and which generate well-described classical geometries [27]. We use the simplest cap geometry, which is obtained for the choice of profile function of the form

$$F_1(v) = a \cos \frac{v}{n_5 R_y} \,, \quad F_2(v) = a \sin \frac{v}{n_5 R_y} \,, \quad F_3(v) = 0 \,, \quad F_4(v) = 0 \,, \tag{2.3}$$

where R_y is the radius of the S^1. The geometry for this choice of $\vec{F}$ had arisen earlier in different studies in [28, 29, 30]. The full solution is:

$$
\begin{aligned}
ds^2 \;\; = \;\; & -\frac{1}{h} \left(dt^2 - dy^2 \right) + hf \left(d\theta^2 + \frac{dr^2}{r^2 + a^2} \right) \\
& + h \left[\left(r^2 + \frac{a^2 Q^2 \cos^2 \theta}{h^2 f^2} \right) \cos^2 \theta d\psi^2 + \left(r^2 + a^2 - \frac{a^2 Q^2 \sin^2 \theta}{h^2 f^2} \right) \sin^2 \theta d\phi^2 \right]
\end{aligned}
$$

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$$-\frac{2aQ}{hf}\left(\cos^2\theta\, dy\, d\psi + \sin^2\theta\, dt\, d\phi\right) + \sum_i R_i^2 dz_i^2\,, \tag{2.4}$$

$$C_{ty}^{(2)} = -\frac{Q}{Q+f}\,, \qquad\qquad C_{t\psi}^{(2)} = -\frac{Qa\cos^2\theta}{Q+f}\,,$$

$$C_{y\phi}^{(2)} = -\frac{Qa\sin^2\theta}{Q+f}\,, \qquad C_{\phi\psi}^{(2)} = Q\cos^2\theta + \frac{Qa^2\sin^2\theta\cos^2\theta}{Q+f} \tag{2.5}$$

where

$$a = \frac{Q}{R_y}\,, \qquad f = r^2 + a^2\cos^2\theta\,, \qquad h = 1 + \frac{Q}{f}\,. \tag{2.6}$$

We take R_y to be large compared to the curvature radius $\sqrt{Q}$ of the cap:

$$\epsilon = \frac{\sqrt{Q}}{R_y} \ll 1\,. \tag{2.7}$$

More precisely, we assume ϵ is small enough that we get several units of the AdS radius between the cap and the neck. As an aside, to see how small ϵ should be we note that the radial part of the metric is

$$ds^2 \approx Q\frac{dr^2}{r^2} = Q(d\log r)^2\,. \tag{2.8}$$

The range of the AdS throat region is from $r \sim a$ to $r \sim \sqrt{Q}$. Each proper length $\sqrt{Q}$ in the r direction accounts for one AdS radius. We then get N_{AdS} units of the AdS radius between the cap and the neck, where

$$N_{AdS} \sim \log\frac{\sqrt{Q}}{a} = \log\frac{R_y}{\sqrt{Q}} = \log\frac{1}{\epsilon} \gg 1 \tag{2.9}$$

for sufficiently small ϵ. Thus if keep only the region $r \ll \sqrt{Q}$, we get a space that is asymptotically $AdS_3 \times S^3 \times T^4$. A similar result follows for all the D1D5 geometries in the limit (2.7) [31]. Thus we can associate a state in the dual CFT with each choice of cap, and also ask about the action of CFT operators $L_{-n}, J_{-n}^a, J_{-n}^i$ on the state.

Let us consider the different parts of the geometry (2.4); this structure is generic for two-charge geometries:

(i) For $r \gg \sqrt{Q}$ we have approximately flat space.

(ii) At $r \sim \sqrt{Q}$ we have the intermediate 'neck' region.

(iii) For $a \ll r \ll \sqrt{Q}$ we have the 'throat', which is the Poincare patch of $AdS_3 \times S^3 \times T^4$.

(iv) At $r \sim a$ we have the 'cap', which depends on the choice of the state.

The geometry (2.4) describes the particular state $|0\rangle_R$ in the Ramond sector of the D1D5 CFT which arises from the spectral flow [32] of the NS vacuum $|0\rangle_{NS}$.

2.2 The outer and inner regions of the background geometry

Our goal is to construct perturbations on the geometry (2.4) that correspond to states such as $L_{-n}|0\rangle_R$. We do this by splitting the geometry into two overlapping regions: an 'outer region' and an 'inner region'. We solve the equation for the perturbation in these two regions, and match the two solutions in the domain of overlap.
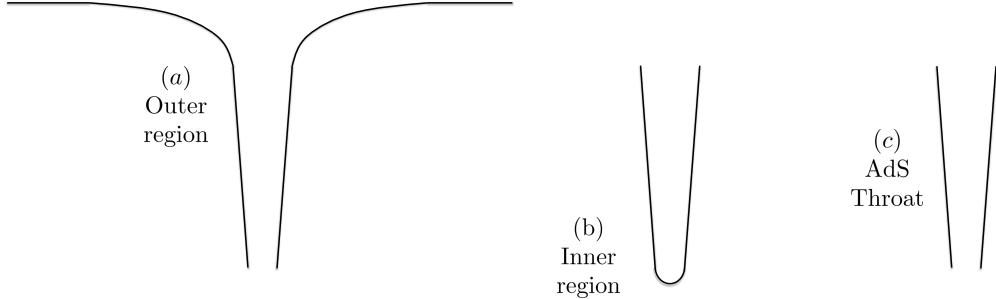

Figure 3: The outer and inner regions, which overlap in the throat region.

2.2.1 The outer region

The outer region is defined as

$$r \gg a . \tag{2.10}$$

The solution (2.4), (2.5) reduces in this limit to the 'naive' D1D5 solution, depicted in Fig. 3 (a):

$$ds^2_{outer} = \left(1 + \frac{Q}{r^2}\right)^{-1} \left[-dt^2 + dy^2\right] + \left(1 + \frac{Q}{r^2}\right) \left[dr^2 + r^2 d\Omega_3^2\right] + \sum_i R_i^2 dz_i^2 , \tag{2.11}$$

$$C^{(2)}_{ty} = -\frac{Q}{Q + r^2}, \quad C^{(2)}_{\psi\phi} = -Q \cos^2 \theta . \tag{2.12}$$

2.2.2 The inner region

The inner region, depicted in Fig. 3 (b), is defined by taking the limit

$$r \ll \sqrt{Q} . \tag{2.13}$$

In this limit the geometry (2.4) becomes

$$
\begin{aligned}
ds^2 \;=\; & -\frac{(r^2 + a^2 \cos^2 \theta)}{Q}(dt^2 - dy^2) + Q\left(d\theta^2 + \frac{dr^2}{r^2 + a^2}\right) + Q(\cos^2 \theta d\psi^2 + \sin^2 \theta d\phi^2) \\
& -2a(\cos^2 \theta dy d\psi + \sin^2 \theta dt d\phi) + \sum_i R_i^2 dz_i^2 .
\end{aligned}
\tag{2.14}
$$

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The change of coordinates

$$\psi_{NS} = \psi - \frac{a}{Q}y, \quad \phi_{NS} = \phi - \frac{a}{Q}t \tag{2.15}$$

brings (2.14) to the form $AdS_3 \times S^3 \times T^4$

$$
\begin{aligned}
ds^2 &= -\frac{(r^2 + a^2)}{Q}dt^2 + \frac{r^2}{Q}dy^2 + Q\frac{dr^2}{r^2 + a^2} \\
&\quad + Q(d\theta^2 + \cos^2\theta d\psi_{NS}^2 + \sin^2\theta d\phi_{NS}^2) + \sum_i R_i^2 dz_i^2 \,.
\end{aligned} \tag{2.16}
$$

2.2.3 The overlap region (throat)

The outer and inner regions overlap in the throat region, defined by

$$a \ll r \ll \sqrt{Q}\,, \tag{2.17}$$

and depicted in Fig. 3 (c). This domain of overlap exists because we have chosen the limit $\epsilon \ll 1$ (2.7). The metric in this region can be obtained by taking the $r \ll \sqrt{Q}$ limit of the outer region solution (2.11), which gives

$$
\begin{aligned}
ds_{thr}^2 &= \frac{r^2}{Q}\left[-dt^2 + dy^2\right] + \frac{Q}{r^2}\left[dr^2 + r^2 d\Omega_3^2\right] + \sum_i R_i^2 dz_i^2\,, \\
C_{ty}^{(2)} &= \frac{r^2}{Q}, \quad C_{\psi\phi}^{(2)} = -Q\cos^2\theta
\end{aligned} \tag{2.18}
$$

where we have added a constant unity to $C_{ty}^{(2)}$ (this does not affect the field strength). This geometry describes the Poincare patch of $AdS_3 \times S^3 \times T^4$.

2.3 The field equations satisfied by the perturbation

The 2-charge solution (2.4), (2.5) is a 10D solution. But since the T^4 has a constant size, dimensional reduction to 6D gives just the metric (2.4) without the torus directions. In this 6D solution, we find that the gauge field is self-dual

$$*_6 F^{(3)} = F^{(3)} \tag{2.19}$$

(We use the sign convention $\epsilon_{tyr\theta\psi\phi} = \sqrt{-g_6}$.)

For the perturbations corresponding to L_{-n}, J_{-n}^a we will take an ansatz that keeps the perturbed $F^{(3)}$ self-dual in the 6D space. The reason for this ansatz is that the 6D theory is described by a set of self-dual fields F_S^r and anti-self-dual fields F_A^i, with the indices r, i belonging to different internal symmetry groups. The background

geometry has only one self-dual field turned on, and there is no natural pairing of any this field with any of the anti-self-dual fields. The operators L_{-n}, J^a_{-n} do not carry any of the indices r, i, so we can expect that only the self-dual field making the background will be involved in the perturbations corresponding to these operators.

For self-dual $F^{(3)}$ we get $F^2 \equiv F^{(3)}_{ABC} F^{(3)ABC} = 0$. If the only contribution to the stress tensor is from $F^{(3)}$, then we get vanishing of the Ricci scalar R. The dilaton can be set to zero. The relevant fields are the metric g_{AB} in 6D and the self dual gauge field $F^{(3)}_{ABC}$. We write $(A, B, \ldots = 1, \ldots, 6)$:

$$g_{AB} = \bar{g}_{AB} + \hat{\epsilon}\, h_{AB}, \qquad C^{(2)}_{AB} = \bar{C}^{(2)}_{AB} + \hat{\epsilon}\, C_{AB} . \tag{2.20}$$

The equations for the perturbation are obtained by expanding to linear order the equations

$$R_{AB} = \frac{1}{4} F^{(3)}_{ACD} F^{(3)CD}_B ,$$
$$*_6 F^{(3)} = F^{(3)} , \qquad F^{(3)}_{ABC} = \partial_A C^{(2)}_{BC} + \partial_B C^{(2)}_{CA} + \partial_C C^{(2)}_{AB} . \tag{2.21}$$

A $U(1)$ current J^i_{-n} involves the torus direction z_i. Let us choose the direction $z_1 \equiv z$. This time we cannot consider just the 6D reduced theory – the perturbation will involve the graviton h_{Az} and the gauge field $C^{(2)}_{Az}$. To linear order in the perturbation, we still have $F^2 = 0$, so that $R = 0$. The equations for the perturbation are obtained from linearizing about the background the 10D equations $(M, N \ldots = 1, \ldots, 10)$

$$R_{MN} = \frac{1}{4} F^{(3)}_{MPQ} F^{(3)PQ}_N ,$$
$$F^{(3)}_{MNP}{}^{;P} = 0, \qquad F^{(3)}_{MNP} = \partial_M C^{(2)}_{NP} + \partial_N C^{(2)}_{PM} + \partial_P C^{(2)}_{MN} . \tag{2.22}$$

2.4   Requirements of the perturbation

We seek a perturbation which satisfies the following criteria:

(i) The perturbation is a pure diffeomorphism in the throat background (2.18). The diffeomorphism at the upper end of the throat (boundary of AdS) should be generated by a vector field corresponding to one of $L_{-n}, J^a_{-n}, J^i_{-n}$.

(ii) The perturbation should *not* be pure gauge in the neck, and should die away at infinity sufficiently rapidly to be normalizable.

(iii) When we continue the solution down the throat, it should match onto a regular solution in the cap background (2.16), as we describe in the next section.

(iv) The perturbation should have a dependence $e^{-in(t-y)/R_y}$, so that it has energy and momentum $E = P = n/R_y$.

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Note that it is not obvious from the differential equations satisfied by the perturbation that such solutions will exist. We have chosen to make the perturbation go over to a diffeomorphism in the throat. With these requirement, the solution can in general go over into a linear combination of growing and decaying solutions at spatial infinity. The fact that it goes over to only the decaying solution follows, of course, from the fact that there is indeed a state with $E = P$ for this physical system.

2.5   The matching process

The outer and inner regions overlap in the throat region

$$a \ll r \ll \sqrt{Q} \tag{2.23}$$

whose metric is given in (2.18). We require the solutions in the outer and inner regions to agree in this overlap region; this will produce a solution that is regular everywhere. This matching process was developed in detail in [20]; here we summarize the main steps.

A perturbation on the geometry (2.4) satisfies a linear differential equation, which we write symbolically as $\tilde{\Box}\phi = 0$. Here ϕ denotes the perturbation, and $\tilde{\Box}$ is a differential operator acting on the perturbation. Suppose we are solving the perturbation in the outer region. In the outer domain $r \gg a$ the full metric (2.4) can be expanded as

$$g = g_{outer} + \epsilon g_1 \tag{2.24}$$

where g_0 is the metric (2.11) and g_1 captures the difference between the full metric g and its approximation g_{outer}. The differential operator $\tilde{\Box}$ can be correspondingly expanded as

$$\tilde{\Box}_g = \tilde{\Box}_{g_{outer}} + \epsilon \Box' \tag{2.25}$$

where $\tilde{\Box}_{g_{outer}}$ is the wave equation for the perturbation in the leading order metric g_{outer} and $\epsilon\Box'$ captures all the subleading effects of the metric correction g_1.

The perturbation itself can now be expanded as

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots \tag{2.26}$$

The equation satisfied by ϕ is

$$\left(\tilde{\Box}_{g_{outer}} + \epsilon\Box'_{outer}\right)\left(\phi_0^{outer} + \epsilon\phi_1^{outer} + \epsilon^2\phi_2^{outer} + \ldots\right) = 0. \tag{2.27}$$

At leading order in ϵ we get

$$\tilde{\Box}_{g_{outer}}\phi_0^{outer} = 0 \tag{2.28}$$

so we just have to solve the wave equation in the geometry (2.11). At next order in ϵ we would have to solve

$$\tilde{\Box}_{g_{outer}}\phi_1^{outer} = -\Box'_{outer}\phi_0^{outer} \tag{2.29}$$

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which is the wave equation with source term $-\Box' \phi_0^{outer}$. This process can be continued in a similar manner to all other orders in ϵ.

Now consider the inner region, and write the wave equation for ϕ^{inner} with a similar expansion to (2.27). The leading order solution satisfies

$$\tilde{\Box}_{g_{inner}} \phi_0^{inner} = 0 \tag{2.30}$$

and we can continue to higher orders in epsilon as before.

Now suppose we have found a solution ϕ_0^{outer} in the outer region that dies off at $r \to \infty$, and a solution ϕ_0^{inner} that is regular in the cap. For a complete solution to exist, it must be possible to choose a normalization for the outer and inner solutions such that

$$\phi_0^{outer} = \phi_0^{inner} + O(\epsilon), \qquad a \ll r \ll \sqrt{Q} \tag{2.31}$$

where $O(\epsilon)$ denotes terms that vanish as $\epsilon \to 0$.

This matching process can be carried out to arbitrarily many orders in ϵ (it was carried out to order ϵ^3 in [20]), but in the present paper we restrict ourselves to noting agreement at the lowest level (2.31).

3 L_{-n} perturbation

3.1 Virasoro algebra and asymptotic diffeomorphisms of AdS_3

We start with a physical discussion of the asymptotic diffeomorphisms of AdS_3 introduced by Brown & Henneaux [9]. We consider the AdS_3 part of the geometry (2.18), dropping the S^3 and T^4 parts for now, and set $Q = 1$, obtaining

$$ds^2 = -r^2 dt^2 + r^2 dy^2 + \frac{dr^2}{r^2} \,. \tag{3.1}$$

The dual CFT is parametrized by the coordinates t, y, and we can imagine it to live at a 'boundary surface' $r = R_{CFT}$.

Let us write $u = t + y, v = t - y$ and make an infinitesimal diffeomorphism

$$v \to v + \hat{\epsilon} e^{-ikv} \tag{3.2}$$

where here and throughout the paper we use $\hat{\epsilon} \ll 1$ as a control of the smallness of the perturbation. Then the AdS_3 metric changes as $g_{\mu\nu} \to g_{\mu\nu} + \hat{\epsilon} h_{\mu\nu}$ with

$$h_{uv} = \frac{ikr^2}{2} e^{-ikv} \,. \tag{3.3}$$

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In the CFT this corresponds to a 'holomorphic diffeomorphism', which changes the 2D metric by a conformal factor.

In the CFT we can cancel this conformal metric change by a Weyl scaling $g_{ij} \to \Omega^2 g_{ij}$. In the dual AdS_3 geometry we can perform the analog of this rescaling by moving the location $r = R_{CFT}$ radially outwards or inwards by an infinitesimal amount, since the metric induced on the 'boundary surface' scales as r^2. Thus we make a diffeomorphism $r \to r + \delta r$ such that

$$\delta g_{uv} = -r\delta r = -\hat{\epsilon} h_{uv} = -\hat{\epsilon}\frac{ikr^2}{2}e^{-ikv} \tag{3.4}$$

which gives

$$\delta r = \hat{\epsilon}\frac{ikr}{2}e^{-ikv} \tag{3.5}$$

and now the metric on the surface $r = R_{CFT}$ is unchanged to leading order. We do however generate a subleading part $h_{vr} = \frac{k^2}{2r}e^{-ikv}$. This can be cancelled by a shift $u \to u - \hat{\epsilon}\frac{k^2}{2r^2}e^{-ikv}$, leaving

$$h_{vv} = -\frac{ik^3}{2}e^{-ikv} . \tag{3.6}$$

The perturbation (3.6) is small enough at large r to be counted among the allowed boundary diffeomorphisms that form the asymptotic symmetry group of AdS_3. (Measured in a local orthonormal frame, it falls as $\frac{1}{r^2}$.)

For our purposes, we normalize the combination of the above diffeomorphisms as follows. The coordinate y in our solution (2.11) has period $2\pi R_y$, so we set $k = \frac{n}{R_y}$, with n an integer. We require the vectors ξ_n^L generating the L_{-n} diffeomorphism to satisfy the commutation relations of the Virasoro algebra (without central charge)

$$\left[\xi_m^L, \xi_n^L\right] = i(m-n)\xi_{m+n}^L . \tag{3.7}$$

In order to do this we normalize the components of ξ_n^L to be

$$\left(\xi_n^L\right)^v = R_y\, e^{-in\frac{v}{R_y}} , \quad \left(\xi_n^L\right)^r = \frac{i}{2}nr\, e^{-in\frac{v}{R_y}} , \quad \left(\xi_n^L\right)^u = \frac{n^2}{2}\frac{Q^2}{r^2 R_y}e^{-in\frac{v}{R_y}} . \tag{3.8}$$

The metric perturbation (3.6) then becomes

$$h_{vv} = -\frac{in^3}{2}\frac{Q}{R_y^2}e^{-in\frac{v}{R_y}} . \tag{3.9}$$

Since we are using an exponential $e^{-in\frac{v}{R_y}}$, we see an overall i in the metric; to obtain the classical fields, one must take the real part of the perturbation. So a diffeomorphism with parameter $\cos\frac{nv}{R_y}$ produces a metric perturbation of the form $\sin\frac{nv}{R_y}$.

Note that the perturbation (3.9) vanishes for the action of the L_0 diffeomorphism but not for L_1 and L_{-1}; this is because we are working in the Poincare patch and

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so are using the asymptotic form (3.8) of the diffeomorphisms rather than the full expressions relevant for global AdS_3 (see e.g. [33]). This is also correct from the point of view of the CFT, where $L_{-1} |\psi\rangle$ should be non-zero for generic Ramond ground states $|\psi\rangle$.

3.2 L_{-n} perturbation in the outer region

As it stands, (3.6) is a just a diffeomorphism of AdS_3, and it is not clear in what sense it will create a new state $L_{-n} |\psi\rangle$. We next present a perturbation that equals the above diffeomorphism in the throat $r \ll \sqrt{Q}$, is *not* a pure diffeomorphism at the neck $r \sim \sqrt{Q}$, and which dies away at infinity. The energy carried by the nontrivial deformation in the neck region will account for the increase in energy expected from the application of an operator L_{-n} to a CFT state.

We construct this perturbation by taking the pure-diffeomorphism perturbation (3.9) and making an ansatz in the full outer region metric (2.11). The ansatz consists of multiplying the perturbation (3.9) by an arbitrary smooth function of r. Solving the field equations (2.21) in the full outer region metric then yields the solution

$$h_{vv} \;\; = \;\; -\frac{in^3}{2}\frac{Q}{R_y^2}\, e^{-in\frac{v}{R_y}}\frac{Q}{Q+r^2}\,, \qquad C \;\; = \;\; 0\,. \tag{3.10}$$

It can be seen that this perturbation is normalizable at infinity: the energy is proportional to $\int d^4x (\partial_r h_{vv})^2$ and so is finite. The perturbation is nontrivial in the neck $r \sim \sqrt{Q}$. In the throat $r \ll \sqrt{Q}$ must match onto a smooth solution in the cap, near $r = 0$. In order to test this, we next study solutions of the field equations in the cap.

3.3 Solution in the inner region

In order to compare a perturbation around the outer background (2.11) and a perturbation around the inner background (2.14), we must take account of the spectral flow transformation (2.15). In Appendix A we show that in each of the perturbations we construct in this paper, this transformation gives only higher order corrections to the matching in the throat, and thus has no effect at the leading order of matching that we do.

The cap geometry we have chosen is global $AdS_3 \times S^3 \times T^4$, given in (2.16). The equations for perturbations around $AdS_3 \times S^3$ were found in [34]. The perturbation we are looking for does not involve the T^4, and will also turn out to be a singlet on the sphere. We take the ansatz for the metric perturbation

$$AdS_3: \quad h_{\mu\nu} = M g_{\mu\nu}\,, \qquad\qquad S^3: \quad h_{ab} = N g_{ab} \tag{3.11}$$

and for the perturbation of the AdS field strength

$$F_{\mu\nu\lambda}^{(3)} = 2(1 + P)\epsilon_{\mu\nu\lambda} \, . \tag{3.12}$$

The gauge field on the sphere remains $F_{abc}^{(3)} = 2\epsilon_{abc}$ since $\int_{S^3} F^{(3)}$ is a fixed number (the charge on the sphere). As shown in [34] and re-derived for completeness in Appendix B, the equations of motion reduce to

$$M + 3N = 0 \, , \qquad P = 2M \, , \qquad (\square - 8)M = 0 \, . \tag{3.13}$$

We seek a solution to $(\square - 8)M = 0$ with the ansatz $M = f(r)e^{-in\frac{v}{R_y}}$ for some function $f(r)$. The solution which is regular at $r = 0$ is

$$M \;=\; A\left(\frac{r^2}{r^2 + a^2}\right)^{\frac{n}{2}}\left(n + 1 + \frac{2r^2}{a^2}\right)e^{-in\frac{v}{R_y}} \tag{3.14}$$

for constant A.

3.4 Matching in the throat region

As discussed in Section 2.5, the solutions in the outer and inner regions must match in the throat region,

$$a \;\ll\; r \;\ll\; \sqrt{Q} \, . \tag{3.15}$$

First consider the inner region solution given by (3.13) and (3.14). Taking the $r \gg a$ limit gives the leading order fields

$$M \;=\; 2A\frac{r^2}{a^2}e^{-in\frac{v}{R_y}} \, , \qquad N \;=\; -\frac{1}{3}M \, , \qquad P \;=\; 2M \, . \tag{3.16}$$

Next consider the outer region solution given by (3.10). We take $r \ll \sqrt{Q}$ and apply a diffeomorphism to match onto the above fields coming from the regular solution in the cap. The diffeomorphism we use is generated by

$$\xi^v = -R_y e^{-in\frac{v}{R_y}} \, , \qquad \xi^r = -\frac{in}{2}e^{-in\frac{v}{R_y}}\frac{Q^2 r}{(Q + r^2)^2} \, , \qquad \xi^u = -\frac{n^2}{2}\frac{Q^2}{r^2 R_y}e^{-in\frac{v}{R_y}} \, . \tag{3.17}$$

Note that the diffeomorphism reduces to $-\xi_n^L$ in the $r \ll \sqrt{Q}$ limit (recall that ξ_n^L generates the L_{-n} diffeomorphism in the AdS throat and is given in (3.8)). Applying this diffeomorphism to the outer region solution (3.10) we get

$$
\begin{aligned}
h_{uv} &= \frac{3in}{2}\frac{r^4}{Q^2}e^{-in\frac{v}{R_y}} \, , & h_{rr} &= 3ine^{-in\frac{v}{R_y}} \, , \\
h_{ab} &= -in\frac{r^2}{Q}e^{-in\frac{v}{R_y}}\bar{g}_{ab} \, , & C_{uv} &= -\frac{3in}{2}\frac{r^4}{Q^2}e^{-in\frac{v}{R_y}}
\end{aligned}
\tag{3.18}
$$

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where $\bar{g}_{ab}$ is the background S^3 metric,

$$\bar{g}_{ab}dx^a dx^b = Q\left[d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2\right].\qquad(3.19)$$

Using the definitions of M, N, P in (3.11), (3.12), we read off the fields

$$M = 3in\frac{r^2}{Q}e^{-in\frac{v}{R_y}}, \qquad N = -\frac{1}{3}M, \qquad P = 2M.\qquad(3.20)$$

So we see that the inner region perturbation (3.16) and the outer region perturbation (3.20) agree if we choose

$$A = \frac{3in}{2}\frac{a^2}{Q} = \frac{3in}{2}\frac{Q}{R_y^2}.\qquad(3.21)$$

3.5 Summary of L_{-n} construction

Let us summarize what we have found above. Suppose we start with the particular 2-charge solution (2.4), (2.5). This describes the D1D5 system in one of the Ramond ground states $|0\rangle_R$. Suppose we wish to derive the gravity solution that corresponds to the state $L_{-n}|0\rangle_R$. Then we add a perturbation that is given for $r \gg a$ by

$$h_{vv} = -in^3\frac{Q}{R_y^2}e^{-in\frac{v}{R_y}}\frac{Q}{Q+r^2}, \qquad C = 0\qquad(3.22)$$

and for $r \ll \sqrt{Q}$ by

$$M = \frac{3in}{2}\frac{Q}{R_y^2}\left(\frac{r^2}{r^2+a^2}\right)^{\frac{n}{2}}\left(n+1+\frac{2r^2}{a^2}\right)e^{-in\frac{v}{R_y}}, \quad N = -\frac{1}{3}M, \quad P = 2M\quad(3.23)$$

where M, N, P are defined in (3.11), (3.12). The expressions (3.22), (3.23) are gauge-equivalent in the overlap region $a \ll r \ll \sqrt{Q}$. Thus together they give a solution that is normalizable at infinity and regular in the cap.

At this stage it may appear that we can take the integer n to be either positive or negative, but only the choice $n > 0$ corresponds to a real excitation. This is because having found the linearized solution to the classical wave equation, we must quantize the perturbation to get the actual allowed excitations. In quantizing a scalar field we write

$$\hat{\phi} = \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega}}[\hat{a}_{\vec{k}}e^{i\vec{k}\cdot\vec{x}-i\omega t} + \hat{a}_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{x}+i\omega t}]\qquad(3.24)$$

and only the function $e^{i\vec{k}\cdot\vec{x}-i\omega t}$ (with $\omega > 0$) represents an allowed excitation. This is, of course, just a restatement of the fact that the vacuum has been selected to be the one that is annihilated by the $\hat{a}_{\vec{k}}$, so that it is the lowest energy state. In our case, the background solution $|0\rangle_R$ is a ground state for the given charges, so quantization of

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excitations will allow only positive energy excitations around it, which behave as $e^{-i\omega t}$ with $\omega > 0$. This implies $n > 0$ in (3.22). The energy of the quantized perturbation is $E = \omega$, which in our case is

$$E = \frac{n}{R_y} \, . \tag{3.25}$$

This is indeed the energy expected from an excitation L_{-n} applied to a CFT state living on a circle of length $2\pi R_y$.

4 Perturbation from sphere diffeomorphism $J^{(3)}_{-n}$

4.1 $SU(2)_L$ algebra on the sphere

The CFT for the D1D5 system has a $\mathcal{N} = 4$ supersymmetric chiral algebra in each of the left and right moving sectors. Thus besides the generators L_{-n} we also have the bosonic generators J^a_{-n} as well as fermionic generators. We now turn to the diffeomorphisms corresponding to the J^a_{-n} (we will not construct the perturbations for the fermionic generators in this paper).

The J^a_{-n} describe a current algebra, which arises from rotations of the sphere. We take the unit sphere S^3 to have the metric

$$ds^2 = d\theta^2 + \cos^2\theta \, d\psi^2 + \sin^2\theta \, d\phi^2 \, . \tag{4.1}$$

The symmetry group of this sphere is $SO(4) \approx SU(2)_L \times SU(2)_R$. The vector fields generating $SU(2)_L$ are

$$
\begin{aligned}
V^{(3)} &= \frac{1}{2}(\partial_\psi - \partial_\phi) \, , \\
V^{(+)} &= \frac{1}{2}e^{i(\psi-\phi)}\Big(\partial_\theta - i\tan\theta\partial_\psi - i\cot\theta\partial_\phi\Big) \, , \\
V^{(-)} &= \frac{1}{2}e^{-i(\psi-\phi)}\Big(\partial_\theta + i\tan\theta\partial_\psi + i\cot\theta\partial_\phi\Big)
\end{aligned}
\tag{4.2}
$$

with commutation relations

$$[V^{(3)}, V^{(+)}] = iV^{(+)}, \quad [V^{(3)}, V^{(-)}] = -iV^{(-)}, \quad [V^{(+)}, V^{(-)}] = 2iV^{(3)} \, . \tag{4.3}$$

From now on, we work with $J^{(3)}_{-n}$ because of the simpler form of $V^{(3)}$ compared to the other vector fields. In particular $V^{(3)}$ generates rigid rotations of the S^3 by the diffeomorphism $\psi \to \psi + \frac{1}{2}\hat{\epsilon}, \quad \phi \to \phi - \frac{1}{2}\hat{\epsilon}$.

In order that the vectors ξ^a_n generating the J^a_{-n} diffeomorphism satisfy the appropriate commutation relations, i.e.

$$
\begin{aligned}
\left[\xi^a_m, \xi^b_n\right] &= i\epsilon^{abc}\,\xi^c_{m+n} \, , & \left[\xi^L_m, \xi^a_n\right] &= -in\,\xi^a_{m+n}
\end{aligned}
\tag{4.4}
$$

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we obtain $\xi_n^a = V^{(a)} e^{-in\frac{v}{R_y}}$, which gives for $\xi_n^{(3)}$:

$$(\xi_n^{(3)})^\psi = \frac{1}{2} e^{-in\frac{v}{R_y}}, \qquad (\xi_n^{(3)})^\phi = -\frac{1}{2} e^{-in\frac{v}{R_y}}. \qquad (4.5)$$

This diffeomorphism applied to the geometry of $AdS_3 \times S^3$ generates a metric perturbation h_{AB} but it also generates a perturbation of the gauge field $C^{(2)}$ since this gauge field is nonzero in the background solution. One can always accompany a diffeomorphism by a gauge transformation

$$C_{AB}^{(2)} \to C_{AB}^{(2)} + \hat{\epsilon}\Big(\partial_A \Lambda_B - \partial_B \Lambda_A\Big) \qquad (4.6)$$

and we will next see that we indeed require such a gauge transformation in order to obtain an ansatz for the perturbation which is regular in the entire geometry (2.11).

4.2 Solution in the outer region

We now consider the diffeomorphism generated by $\xi_n^{(3)}$, given in (4.5), applied to the throat $AdS_3 \times S^3 \times T^4$ geometry (2.18). It generates the perturbation

$$
\begin{aligned}
h_{v\psi} &= -\frac{in}{2}\frac{Q}{R_y} \cos^2\theta\, e^{-in\frac{v}{R_y}}, & C_{v\psi} &= \frac{in}{2}\frac{Q}{R_y} \cos^2\theta\, e^{-in\frac{v}{R_y}}, \\
h_{v\phi} &= \frac{in}{2}\frac{Q}{R_y} \sin^2\theta\, e^{-in\frac{v}{R_y}}, & C_{v\phi} &= \frac{in}{2}\frac{Q}{R_y} \cos^2\theta\, e^{-in\frac{v}{R_y}}.
\end{aligned}
\qquad (4.7)
$$

We will soon extend this to the full geometry (2.11), but as it stands we have a difficulty: after introducing a dependence on r, the component $C_{v\phi}$ will lead to a field strength term $F_{v\phi r}^{(3)}$ which will have an angular dependence $\sim \cos^2\theta$. This is unacceptable, since in a unit orthonormal frame we will have $F_{\hat{r}\hat{v}\hat{\phi}}^{(3)} \sim \frac{\cos^2\theta}{\sin\theta}$, which diverges at $\theta = 0$. Thus we make a gauge transformation (4.6) with

$$\Lambda_\phi = Q\, e^{-in\frac{v}{R_y}} \qquad (4.8)$$

which yields

$$
\begin{aligned}
h_{v\psi} &= -\frac{in}{2}\frac{Q}{R_y} \cos^2\theta\, e^{-in\frac{v}{R_y}}, & C_{v\psi} &= \frac{in}{2}\frac{Q}{R_y} \cos^2\theta\, e^{-in\frac{v}{R_y}}, \\
h_{v\phi} &= \frac{in}{2}\frac{Q}{R_y} \sin^2\theta\, e^{-in\frac{v}{R_y}}, & C_{v\phi} &= -\frac{in}{2}\frac{Q}{R_y} \sin^2\theta\, e^{-in\frac{v}{R_y}}.
\end{aligned}
\qquad (4.9)
$$

To extend this to a perturbation in the full outer region (2.11), we take an ansatz where we multiply each of the above terms by a smooth function of r. The field equations (2.21) yield the solution

$$
h_{v\psi} = -\frac{in}{2}\frac{Q}{R_y} \cos^2\theta\, e^{-in\frac{v}{R_y}} \frac{Q}{Q+r^2}, \qquad C_{v\psi} = \frac{in}{2}\frac{Q}{R_y} \cos^2\theta\, e^{-in\frac{v}{R_y}} \frac{Q}{Q+r^2},
$$

$$h_{v\phi} = \frac{in}{2}\frac{Q}{R_y}\sin^2\theta\, e^{-in\frac{v}{R_y}}\frac{Q}{Q+r^2}\,, \qquad C_{v\phi} = -\frac{in}{2}\frac{Q}{R_y}\sin^2\theta\, e^{-in\frac{v}{R_y}}\frac{Q}{Q+r^2}\,. \quad (4.10)$$

This solution is normalizable at infinity, and nontrivial in the neck $r \sim \sqrt{Q}$. In the throat $r \ll \sqrt{Q}$ the solution matches to a solution in the inner region as we shall see in Section 4.4.

4.3 Solution in the inner region

In the inner region $r \ll \sqrt{Q}$, we make the ansatz

$$h_{\mu a}(x,y) = K_\mu(x)Y_a(y)\,, \qquad C_{\mu a}(x,y) = Z_\mu(x)Y_a(y) \qquad (4.11)$$

where as before μ is an AdS_3 coordinate and a is an S^3 coordinate. We must choose the spherical harmonic corresponding to the diffeomorphism (4.5), i.e.

$$Y_\theta = 0\,, \quad Y_\psi = \cos^2\theta\,, \quad Y_\phi = -\sin^2\theta\,. \qquad (4.12)$$

In Appendix C.1 we study the field equations (2.21) under this ansatz and show that they yield the following solution which is regular in the cap. Z_μ is given by

$$
\begin{aligned}
Z_v &= C\,a\Big(\frac{r^2}{r^2+a^2}\Big)^{\frac{n}{2}}\Big(n+1+\frac{2r^2}{a^2}\Big)e^{-in\frac{v}{R_y}}\,,\\
Z_u &= C\,a\Big(\frac{r^2}{r^2+a^2}\Big)^{\frac{n}{2}}e^{-in\frac{v}{R_y}}\,,\\
Z_r &= C\,in\frac{a^2Q}{r(r^2+a^2)}\Big(\frac{r^2}{r^2+a^2}\Big)^{\frac{n}{2}}e^{-in\frac{v}{R_y}}
\end{aligned}
\qquad (4.13)
$$

for constant C, and K_μ is given by

$$K_\mu = -2Z_\mu\,. \qquad (4.14)$$

We next show that the perturbations in the inner and outer regions are gauge equivalent in the throat region.

4.4 Matching in the throat region

We now examine the solutions in the throat region

$$a \ll r \ll \sqrt{Q}\,. \qquad (4.15)$$

In relating the inner and outer regions we have to use the spectral flow transformation (2.15), but as we show in Appendix A the effect of this only arises at higher orders in perturbation theory.

We first take the $r \gg a$ limit in the inner region solution (4.13), which gives the leading order fields

$$K_v = -4C\frac{r^2}{a}e^{-in\frac{v}{R_y}}, \qquad Z_v = 2C\frac{r^2}{a}e^{-in\frac{v}{R_y}}. \qquad (4.16)$$

Using (4.12) this corresponds to

$$\begin{aligned}
h_{v\psi} &= -4C\frac{r^2}{a}\cos^2\theta\,e^{-in\frac{v}{R_y}}, & C_{v\psi} &= 2C\frac{r^2}{a}\cos^2\theta\,e^{-in\frac{v}{R_y}}, \\
h_{v\phi} &= 4C\frac{r^2}{a}\sin^2\theta\,e^{-in\frac{v}{R_y}}, & C_{v\phi} &= -2C\frac{r^2}{a}\sin^2\theta\,e^{-in\frac{v}{R_y}}.
\end{aligned} \qquad (4.17)$$

We next take $r \ll \sqrt{Q}$ in the outer region solution (4.10) and apply the diffeomorphism and gauge transformation generated by

$$\xi_n^\psi = -e^{-in\frac{v}{R_y}}, \qquad \xi_n^\phi = e^{-in\frac{v}{R_y}}, \qquad \Lambda_\phi = -e^{-in\frac{v}{R_y}} \qquad (4.18)$$

which is the reverse of the $J_{-n}^{(3)}$ perturbation we started with, given in (4.5) and (4.8). The resulting perturbation for $r \ll \sqrt{Q}$ has the form

$$\begin{aligned}
h_{v\psi} &= i\frac{r^2}{R_y}\cos^2\theta\,e^{-in\frac{v}{R_y}}, & C_{v\psi} &= -\frac{i}{2}\frac{r^2}{R_y}\cos^2\theta\,e^{-in\frac{v}{R_y}}, \\
h_{v\phi} &= -i\frac{r^2}{R_y}\cos^2\theta\,e^{-in\frac{v}{R_y}}, & C_{v\phi} &= \frac{i}{2}\frac{r^2}{R_y}\cos^2\theta\,e^{-in\frac{v}{R_y}}.
\end{aligned} \qquad (4.19)$$

We see that we get agreement between the inner region solution (4.17) and the outer region solution (4.19) if we choose

$$C = \frac{i}{4}\frac{a}{R_y} = \frac{i}{4}\frac{Q}{R_y^2}. \qquad (4.20)$$

5 Perturbation from torus diffeomorphisms J_{-n}^i

We next construct the perturbation arising from the J_n^i diffeomorphisms, proceeding analogously to the previous sections. This perturbation is technically very similar to the perturbation from the sphere diffeomorphisms; the main difference between the two cases comes from the fact that S^3 has curvature, while T^4 does not. Since the T^4 is flat, the z_i do not mix, and we can let the z_i index lie along any one of the torus cycles, say $z_1 \equiv z$.

The J_{-n}^i do not have any non-trivial commutation relations amongst themselves, there is simply the commutation relation with the L_{-n} generators. The vector ξ_n^z generating the J_{-n}^z diffeomorphism must then obey

$$\left[\xi_m^L, \xi_n^z\right] = -in\,\xi_{m+n}^z, \qquad (5.1)$$

where ξ_m^L is defined in (3.8), so we take ξ_n^z to be

$$\xi_n^z = e^{-in\frac{v}{R_y}}. \qquad (5.2)$$

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5.1 Solution in the outer region

Following the same procedure as before, we apply the diffeomorphism (5.2) to the throat background (2.11). This gives the perturbation

$$h_{vz} = -in\frac{R_z^2}{R_y}e^{-in\frac{v}{R_y}}. \qquad (5.3)$$

As we did for the case of $J_{-n}^{(3)}$, we also perform a gauge transformation $C^{(2)} \to C^{(2)} + d\Lambda^{(1)}$, choosing here

$$\Lambda_z^{(1)} = -R_z^2\, e^{-in\frac{v}{R_y}}. \qquad (5.4)$$

This generates the field

$$C_{vz} = in\frac{R_z^2}{R_y}e^{-in\frac{v}{R_y}}. \qquad (5.5)$$

To find a solution that is not pure gauge everywhere we assume an ansatz where we multiply the fields h_{vz}, C_{vz} above by a smooth function of r. The field equations yield the solution

$$h_{vz} = -in\frac{R_z^2}{R_y}\, e^{-in\frac{v}{R_y}}\frac{Q}{Q+r^2}, \qquad C_{vz} = in\frac{R_z^2}{R_y}\, e^{-in\frac{v}{R_y}}\frac{Q}{Q+r^2}. \qquad (5.6)$$

As before, this solution is normalizable at infinity and in the throat $r \ll \sqrt{Q}$ it matches on to a smooth solution in the cap, as we show in Section 5.3.

5.2 Solution in the inner region

In the inner region we use the following notation for the torus perturbation:

$$h_{\mu z} = K_\mu, \qquad C_{\mu z} = Z_\mu. \qquad (5.7)$$

In Appendix D we find that the field equations (2.22) yield the solution

$$\begin{aligned}
Z_v &= D\,a\Big(\frac{r^2}{r^2+a^2}\Big)^{\frac{n}{2}}\Big(n+1+\frac{2r^2}{a^2}\Big)e^{-in\frac{v}{R_y}}, \\
Z_u &= D\,a\Big(\frac{r^2}{r^2+a^2}\Big)^{\frac{n}{2}}e^{-in\frac{v}{R_y}}, \\
Z_r &= D\,in\frac{a^2 Q}{r(r^2+a^2)}\Big(\frac{r^2}{r^2+a^2}\Big)^{\frac{n}{2}}e^{-in\frac{v}{R_y}}
\end{aligned} \qquad (5.8)$$

for constant D, with K given by

$$K_\mu = -Z_\mu. \qquad (5.9)$$

Note that the relation between K and Z is the main difference between the inner region solutions coming from the sphere diffeomorphism and those coming from the torus diffeomorphism. In the sphere case, we have $K_\mu = -2Z_\mu$. The difference can be traced to the fact that the sphere has curvature, unlike the torus.

5.3 Matching in the throat region

We now examine the solutions in the throat region

$$a \ll r \ll \sqrt{Q}. \tag{5.10}$$

We first take the $r \gg a$ limit in the inner region solution (5.8), which gives the leading order fields

$$K_v = -2D \frac{r^2}{a} e^{-in\frac{v}{R_y}}, \qquad Z_v = 2D \frac{r^2}{a} e^{-in\frac{v}{R_y}}. \tag{5.11}$$

Using (5.7) this corresponds to

$$h_{vz} = -2D \frac{r^2}{a} e^{-in\frac{v}{R_y}}, \qquad C_{vz} = 2D \frac{r^2}{a} e^{-in\frac{v}{R_y}}. \tag{5.12}$$

We next take $r \ll \sqrt{Q}$ in the outer region solution (5.6) and apply the diffeomorphism and gauge transformation generated by

$$\xi^z = -e^{-in\frac{v}{R_y}}, \qquad \Lambda_z^{(1)} = R_z^2 e^{-in\frac{v}{R_y}} \tag{5.13}$$

which is the reverse of the J_{-n}^z perturbation we started with, given in (5.2), (5.4). The resulting perturbation for $r \ll \sqrt{Q}$ has the form

$$h_{vz} = in \frac{R_z^2}{R_y} \frac{r^2}{Q} e^{-in\frac{v}{R_y}}, \qquad C_{vz} = -in \frac{R_z^2}{R_y} \frac{r^2}{Q} e^{-in\frac{v}{R_y}}. \tag{5.14}$$

We see that (5.14) agrees with (5.12) if we choose

$$D = -\frac{i}{2} \frac{a R_z^2}{R_y Q} = -\frac{i}{2} \frac{R_z^2}{R_y^2}. \tag{5.15}$$

6   A cross-check of our results

In the preceding sections we have found approximate perturbations around the background geometry (2.4) and shown that the perturbations have the right properties to be dual to states created by generators of the chiral algebra of the D1-D5 CFT.

Our identification of states is based on [9]. Let us ask if we can obtain a cross-check on this identification using general results from the AdS/CFT correspondence. Since the background geometry (2.4) is asymptotically flat, to carry out such a check we must decouple the *AdS* region. A test will then be provided by calculating the one-point functions of chiral primaries from the gravity and CFT sides [35]. Such tests have been done for the spectral flow coordinate transformation [36] and for general two-charge D1-D5 geometries [37].

In this section we carry out such a test, and exhibit the required agreements. Thus our construction based on [9] is consistent with the above-mentioned *AdS*/CFT work. We follow in places [36, 37].

6.1 The L_{-n} perturbation

We first derive the one-point function for the stress-energy tensor in the L_{-n} perturbation. On the CFT side, consider the state

$$|\mu\rangle \;=\; (\mathbb{1} + \mu L_{-n})\,|0\rangle_R \tag{6.1}$$

where we have introduced the (complex) parameter μ. By the standard relation between classical perturbations and coherent states, μ will be proportional to $\hat{\epsilon}$, the parameter controlling the size of the classical perturbation; thus we work to linear order in μ. The above state is proposed to be dual to the perturbed metric constructed in Section 3.

In this state, to linear order in μ the only nonzero expectation values are those for the operators $L_{\pm n}$. The mode L_n has expectation value

$$\langle \mu|\, L_n\, |\mu\rangle \;=\; \mu \frac{n_1 n_5}{2} n^3 \tag{6.2}$$

where we have used

$$[L_m,\ L_n] \;=\; \frac{c}{12}(m^3 - m)\delta_{m+n,0} + (m-n)L_{m+n}\,, \qquad c = 6n_1 n_5\,. \tag{6.3}$$

The mode L_{-n} has an analogous one-point function; since have written our classical perturbation in the form $e^{-in\frac{v}{R_y}}$ (and not $\cos(n\frac{v}{R_y})$) we shall deal only with L_n. The energy L_0 of the perturbation appears at quadratic order in μ, and so is beyond the scope of this paper.

On the gravity side, the constant mode at the asymptotically AdS part of the inner region is the part generated by the L_{-n} diffeomorphism, which from (3.9) is

$$h_{vv} \;=\; -\frac{in^3}{2}\frac{Q}{R_y^2}\, e^{-in\frac{v}{R_y}}\,. \tag{6.4}$$

To compare to the CFT stress tensor, we make the Fefferman-Graham expansion using the coordinate $z = Q/r$. We temporarily use μ, ν for the remaining two AdS_3 directions. The expansion is

$$ds^2 \;=\; \frac{Q}{z^2}\Big[dz^2 + \Big(g_{(0)\mu\nu} + z^2 g_{(2)\mu\nu} + \cdots\Big)dx^\mu dx^\nu\Big]\,, \tag{6.5}$$

where '$\cdots$' indicates terms which do not contribute to our checks. Using complex coordinates[1], the one-point function for the holomorphic stress-energy tensor is given at linear order in $\hat{\epsilon}$ by (see e.g. [37])

$$\langle T_{ww}\rangle \;=\; n_1 n_5 \Big(g_{(2)ww} + \cdots\Big)$$

---

[1] We define the coordinates $\sigma = \frac{y}{R_y}, \tau = \frac{t}{R_y}$ so that σ has period 2π. We identify the CFT base space coordinates as σ, τ, and in terms of the Euclidean time $\tau_E = i\tau$, we define $w = \sigma + i\tau_E = \sigma - \tau$.

$$= -\hat{\epsilon} \, \frac{in^3}{2} (n_1 n_5) \, e^{inw} \, . \tag{6.6}$$

This implies that the mode L_n has the one-point function

$$\langle L_n \rangle = \frac{1}{2\pi} \oint dw \, e^{-inw} \langle T_{ww} \rangle = -\hat{\epsilon} \, i \, n^3 \, \frac{n_1 n_5}{2} \, . \tag{6.7}$$

Since μ is proportional to $\hat{\epsilon}$, this is consistent with the CFT one-point function (6.2).

6.2   The $J_{-n}^{(3)}$ and J_{-n}^z perturbations

We now turn to the $J_{-n}^{(3)}$ and J_{-n}^z perturbations, and for concreteness we work with $J_{-n}^{(3)}$. At the asymptotically AdS part of the inner region, both the $J_{-n}^{(3)}$ and J_{-n}^z perturbations have constant terms of the form $h_{AB} = -C_{AB}$. As a result, we expect all the steps to go through also for the J_{-n}^z perturbation.

On the CFT side, consider the state

$$|\lambda\rangle = \left(\mathbb{1} + \lambda J_{-n}^{(3)} \right) |0\rangle_R \, . \tag{6.8}$$

In this state, to linear order in λ the only nonzero expectation values are those for the operators $J_{\pm n}^{(3)}$. Again we deal only with $J_n^{(3)}$ which has expectation value

$$\langle \lambda | \, J_n^{(3)} \, | \lambda \rangle = \lambda \frac{n_1 n_5}{2} n \tag{6.9}$$

where we have used

$$[J_m^a, \, J_n^b] = \frac{c}{12} m \delta^{ab} \delta_{m+n,0} + i \epsilon^{ab}{}_c J_{m+n}^c \, , \qquad c = 6 n_1 n_5 \, . \tag{6.10}$$

On the gravity side, we set up some notation for reading off the one-point functions. Following our conventions for the $SU(2)$ generators (4.2), we form the combinations

$$g_\mu^{(3)} = \frac{1}{2} \left(g_\mu{}^\psi - g_\mu{}^\phi \right) , \qquad g_\mu^{(\bar{3})} = \frac{1}{2} \left(g_\mu{}^\psi + g_\mu{}^\phi \right) \tag{6.11}$$

and similarly for $C^{(2)}$. We then define the one-forms $A^{(3)}$, $A^{(\bar{3})}$, by

$$A_\mu^{(3)} = -\left[g_\mu^{(3)} - (C^{(2)})_\mu^{(3)} \right] , \qquad A_\mu^{(\bar{3})} = -\left[g_\mu^{(\bar{3})} + (C^{(2)})_\mu^{(\bar{3})} \right] \, . \tag{6.12}$$

We now focus on the holomorphic $SU(2)_L$; analogous expressions apply for $SU(2)_R$. We then make the Fefferman-Graham expansion

$$A^{(3)} = \mathcal{A}^{(3)} + z^2 A_{(2)}^{(3)} + \cdots \tag{6.13}$$

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The one-point function for $J^{(3)}$ is then given by (see e.g. [37])

$$\langle J^{(3)} \rangle = \frac{n_1 n_5}{2} \mathcal{A}^{(3)} \,. \tag{6.14}$$

For the case at hand the constant mode at the asymptotically AdS part of the inner region is the part generated by the $J_{-n}^{(3)}$ diffeomorphism. From (4.9), at order $\hat{\epsilon}$ we obtain[2]

$$\mathcal{A}^{(3)} = -\hat{\epsilon} \, in \, e^{inw} dw \qquad \Rightarrow \qquad \langle J^{(3)} \rangle = -\hat{\epsilon} \, in \frac{n_1 n_5}{2} e^{inw} dw \,. \tag{6.15}$$

This implies that the mode $J_n^{(3)}$ has the one-point function

$$\langle J_n^{(3)} \rangle = \frac{1}{2\pi} \oint dw \, e^{-inw} \langle J_w^{(3)} \rangle = -\hat{\epsilon} \, in \frac{n_1 n_5}{2} \,. \tag{6.16}$$

Since λ is proportional to $\hat{\epsilon}$, this is consistent with the CFT one-point function (6.9).

7 Discussion

7.1 Summary of our results

Let us summarize our construction. We have wrapped n_5 D5 branes on $T^4 \times S^1$ and n_1 D1 branes on S^1. Their bound state has $\exp[2\sqrt{2}\pi\sqrt{n_1 n_5}]$ degenerate ground states $|\psi_i\rangle_R$, where the subscript R denotes the Ramond sector. We take the S^1 radius to be large compared to the AdS curvature length scale; then the low energy dynamics of the D1D5 bound state gives a 1+1 dimensional CFT along t, y, where y parametrizes the S^1.

The generic D1D5 bound state is quantum in nature and the gravitational field it sources is not a priori well-described by a classical geometry all the way into the cap region. Certain states do however have backreactions well-described by classical geometries (for a discussion see e.g. [31]); let $|\psi\rangle_R$ be such a state. Previous work has taught us that these geometries have the structure of flat space at infinity, then an intermediate 'neck' region, then a 'throat' which is approximately the Poincare patch of $AdS_3 \times S^3 \times T^4$, and a cap whose structure depends on the state $|\psi\rangle_R$.

Since we can describe the D1D5 system as a CFT, we can ask for the gravity description of a state such as $L_{-n}|\psi\rangle_R$, which carries D1, D5 and P charge. Following the general ideas of [9] we expect that in this AdS region the L_{-n} operators are represented by diffeomorphisms that leave the throat asymptotically $AdS_3 \times S^3 \times T^4$.

---

[2]Rather than introduce additional notation, we leave it as understood that the parameter $\hat{\epsilon}$ may not in general be the same between the different perturbations.

We can of course continue this diffeomorphism out past the neck in any way we like, but a pure diffeomorphism of the starting geometry will not give the gravitational solution created by $L_{-n}|\psi\rangle_R$, since the required solution must have more energy and momentum than the solution for the state $|\psi\rangle_R$.

Taking an ansatz $e^{-in\frac{(t-y)}{R_y}}$ to account for the energy and momentum, we find solutions that reduce to diffeomorphisms in the throat, but are *not* pure gauge in the neck. Our solutions are normalizable at infinity, and we match them to regular solutions in the particular cap we choose, namely the cap for the state obtained from spectral flow of the NS vacuum.

Proceeding in this manner, we obtain perturbations corresponding to the generators of the chiral algebra $L_{-n}, J^{(3)}_{-n}$ in the D1D5 CFT, as well as for the four chiral $U(1)$ currents J^i_{-n} arising from translation symmetry along the four directions of T^4. As noted in Section 3.5, upon quantization only the perturbations with $n > 0$ correspond to physical excitations.

Since the nontrivial part of the perturbation is concentrated at the neck which forms a natural boundary to the AdS region, it is plausible that the states we find are in some way related to the 'singleton' (or 'doubleton') representations that lie at the boundary of AdS [38], see also [39].[3] Note however that the neck is not part of AdS itself, and it breaks the rotational $SO(4)$ symmetry of the S^3. This happens because the physical D1D5 states are in the Ramond (R) sector, while the global $AdS_3 \times S^3$ geometry describes the NS vacuum. The relation between R and NS sector states is given by a 'spectral flow' (2.15) which uses a choice of direction in the S^3.

A consequence of this breaking of symmetry is that while L_0, L_1, L_{-1} are isometries of global $AdS_3 \times S^3$, they differ in their effect on the physical R sector state given by the solution (2.4). L_0 is still a symmetry of the R sector solution, so it generates no perturbation, but L_{-1} is *not* a symmetry and does generate a nontrivial perturbation; this is of course in agreement with the fact that $L_{-1}|\psi\rangle_R \neq 0$ on R sector states. (L_1 is a annihilation operator that kills the L_{-1} perturbation after quantization).

7.2   Relation to previous attempts to construct 'hair'

In this paper we have considered linear perturbations, with the metric and gauge field given by $g_{AB} = \bar{g}_{AB} + \hat{\epsilon} h_{AB}, C^{(2)}_{AB} = \bar{C}^{(2)}_{AB} + \hat{\epsilon} C_{AB}$. It turns out however that in the outer region we can set $\hat{\epsilon}$ to be arbitrary, and obtain an exact solution of the field equations. These nonlinear solutions correspond to CFT operators like $\exp[\hat{\epsilon} L_{-n}]$ where $\hat{\epsilon}$ is no longer infinitesimal.

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[3]In [40] an approximate computation of a singleton boundary mode was given for the NS5 brane geometry.

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In the case of the $U(1)$ currents in the torus directions, the nonlinear solutions in the outer region were previously found and studied in in [21] for deformations of the black string. These solutions were, later found to be singular at the horizon [22]. Such a singularity can be expected because of the general idea of the 'no hair theorem', namely that it is difficult to have any smooth perturbation of a horizon.

On a similar note, in [23] the extremal D1D5P black hole geometry (with horizon) was considered, and linearized perturbations around this geometry were constructed with support in the neck. In a following paper [24] it was found that the perturbations were singular at nonlinear order at the horizon, for the same reasons found in [22].

In our construction, however, there is no such singularity; the background geometry does allow the perturbations we construct. It is interesting to note the differences between the solutions of [21, 23] and those of this paper.

1. The geometries of [21, 23] had horizons, while we have a 'cap'; thus our perturbations do not suffer from this singularities mentioned above. However we are required to impose regularity in the cap, which we do.[4]

2. For the purposes of [23] the perturbations were not required to exist as states, since they were only used to relate the state counts of the 5D black hole and the 4D black hole. They turn out to be singular at the horizon in both cases, and so drop out of both sides of the state count relation. In our case the perturbations correspond to states like $L_{-n}|\psi\rangle_R$, so it would be strange if they did not exist for the full asymptotically flat D1D5 geometry created by the branes.

3. On a technical note, the perturbations in the R^4 directions in [21, 23] are in an $l = 1$ spherical harmonic while the closest analog for our case, the L_{-n} perturbations, are in the $l = 0$ harmonic.

We note in passing that in [23, 24] the term 'hair' was used for degrees of freedom living far away from an event horizon. This appears different, however, from the traditional terminology of 'hair' used by relativists, where 'hair' is used for perturbations *at* the horizon. In this sense [21, 22] were attempts to construct 'hair', while in [23, 24] and this paper there is a long AdS throat separating the perturbation from the cap or horizon. The long AdS throat is necessary so that one may apply the tools of AdS/CFT, and its length is of order $\log \frac{1}{\epsilon}$ where $\epsilon = \frac{\sqrt{Q}}{R_y}$, as discussed in Section 2.1.

---

[4]We examine only linear perturbations of the geometry; this is adequate since a regular perturbation on a smooth solution is not expected to be singular at nonlinear order.

7.3 Future directions

The boundary diffeomorphisms correspond to the application of the chiral algebra generators $L_{-n}, J_{-n}^{(3)}, J_{-n}^z$ to states $|\psi_i\rangle_R$. The space of states we can generate this way corresponds to a central charge $c = 6$; this is of course in accord with the fact that in AdS/CFT duality a $U(1)$ center of mass degree of freedom lives at the boundary of AdS while the remaining states (corresponding to $c = 6(n_1 n_5 - 1)$) lie near $r = 0$. In this sense the construction of this paper directly address only a very small family of microstates. But the construction also shows us a way of understanding states that do *not* live at the boundary, as we now discuss.
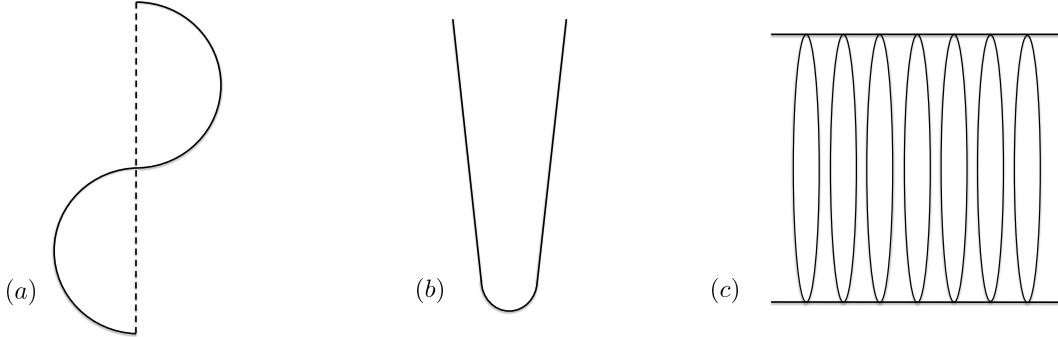
Figure 4: The background geometry and corresponding CFT state used in this paper.

The geometries for the two-charge extremal D1D5 states $|\psi_i\rangle_R$ are generated by taking a vibration profile for the string in the NS1P duality frame, finding the metric for this profile, and then dualizing to the D1D5 frame. When the vibration profile was a single turn of a helix (2.3), depicted in Fig. 4 (a), we obtained global $AdS_3 \times S^3$ for the D1D5 geometry 4 (b). The corresponding D1D5 state $|0\rangle_R$ is described in the orbifold CFT [41, 42] by the completely untwisted sector, depicted in Fig. 4 (c). On this state we have

$$L_{-n}|0\rangle_R = \sum_{k=1}^{n_1 n_5} L_{-n}^{(k)} \left(\prod_k |0_k\rangle_R \right) \tag{7.1}$$

where the index k labels the $n_1 n_5$ untwisted copies of the $c = 6$ CFT which make up the state $|0\rangle_R$, and $L_{-n}^{(k)}$ is the operator L_{-n} acting on the kth copy. Thus we get an action of the *diagonal* L_{-n} arising from the set of individual $L_{-n}^{(k)}$.

Now consider a state $|\psi\rangle_R$ generated by the NS1P profile in Fig. 5 (a). The corresponding D1D5 geometry 5 (b) has a small 'subthroat' corresponding to the high frequency part of the NS1P profile, and in the orbifold CFT we obtain a state with a set of twisted strings 5 (c). Since the subthroat is small, it opens into a part of the global AdS that is close to flat space on the scale of the subthroat. Then the computations done in this paper tell us that in a suitable approximation one may construct modes localized at the neck of the subthroat. It is natural to identify an
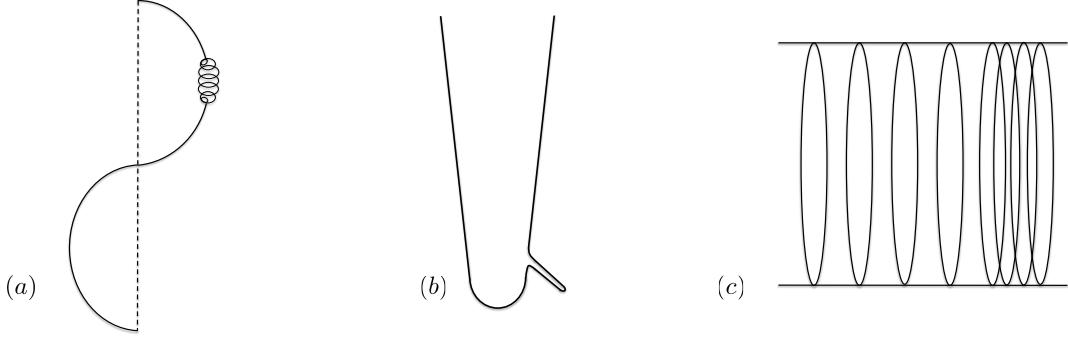
Figure 5: Branching throat with one small subthroat.

L_{-n} perturbation of the subthroat with the state

$$|0_1\rangle_R \otimes |0_2\rangle_R \ldots |0_n\rangle_R \otimes \sum_{k=1}^{K} L_{-n}^{(k)} \left(\prod_{k=1}^{K} |0_k^s\rangle_R \right) \tag{7.2}$$

where now the L_{-n} operators have been applied to the set of twisted copies $|0_k^s\rangle_R$ (s denotes the order of the twist), and $k = 1, \ldots K$ runs over the different twisted copies.

Thus we see that the perturbations localized at the neck leading to flat space correspond to the diagonal action of L_{-n} on all copies of the CFT, while similar modes located inside the 'cap' region correspond to applying L_{-n} *unequally* to the different copies of the CFT.
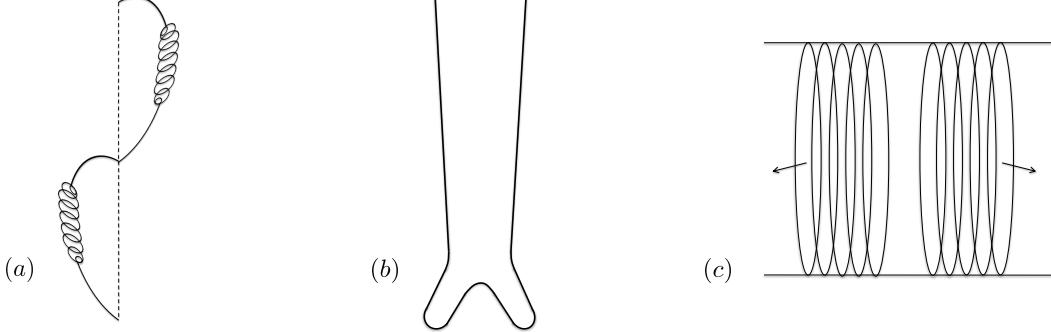
Figure 6: Branching throat with two large subthroats.

As another example, consider the branching throat geometry with two large subthroats depicted in Fig. 6 (b). For each subthroat we can look for a perturbation of the kind we have constructed, and we then look at the antisymmetric combination of these perturbations. This combination will cancel out as we go far above the junction region of the throats, and so the perturbation will be localized around this junction

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region itself. The corresponding CFT state would have the structure

$$\left(\sum_{k=1}^{\frac{n_1 n_5}{2}} L_{-n}^{(k)} - \sum_{k=\frac{n_1 n_5}{2}+1}^{n_1 n_5} L_{-n}^{(k)} \right) \left(\prod_{k=1}^{\frac{n_1 n_5}{2}} |0_k^+\rangle_R \otimes \prod_{k=\frac{n_1 n_5}{2}+1}^{n_1 n_5} |0_k^-\rangle_R \right) \tag{7.3}$$

where the 2-charge extremal state corresponding to the branched-throat geometry has half the copies of the $c = 6$ CFT in one spin state $|0_k^+\rangle_R$ and half in the opposite spin state $|0_k^-\rangle_R$. We see that the action of L_{-n} is not the diagonal one that would give a mode localized at the neck leading to asymptotic infinity, but one where we apply the L_{-n} with different signs to the two sets of CFT copies.

We hope to return for a more detailed analysis of such states elsewhere, but for now we just note that such states at the cap are examples of the 'hair' which should replace the naive horizon geometry of the traditional black hole in the fuzzball picture. In this picture there are no states with a regular horizon, and the above examples illustrate how some specific classes of CFT states may look in the gravity picture.

It would also be interesting to explore the types of perturbations considered in this paper in other setups in which there is a black hole with a locally AdS_3 throat, such as the extremal, vanishing horizon (EVH) black holes that have received interest recently [43], and to investigate relations to the elliptic genus of AdS_3 [44, 45]. One could also attempt to construct similar perturbations in AdS spacetimes of different dimensions; while there is a full Virasoro algebra only for AdS_3, we expect there to be perturbations corresponding to symmetry generators of the global symmetry groups of AdS spacetimes of different dimensions.

An obvious further avenue is to carry out the construction in the present paper to next order in perturbation theory, and to investigate whether this can be generalized to an all-order solution. We plan to return to this in a future paper.

Acknowledgements

A Spectral flow

We have split the full geometry (2.4) into two parts: an outer region (2.11) and an inner region (2.14). The angular coordinates on the S^3 are related by the spectral flow [32] map given in (2.15),

$$\psi_{NS} \;=\; \psi - \frac{a}{Q}y, \quad \phi_{NS} \;=\; \phi - \frac{a}{Q}t\,. \tag{A.1}$$

We solve the wave equation in the outer region, then in the inner region, and then match the two solutions. For this matching, we should take into account the above transformation relating the two coordinates. But as we will now see, to the order where we are working, this change is *not* significant. If we wished to work to higher orders in the small parameter ϵ defined in (2.7) then we would have to take (A.1) into account; this was done for example in [20] to several orders in ϵ.

Suppose a vector V_A^{inner} is given in the inner region coordinates, and we wish to find V_A^{outer}, the components in the outer region coordinates. The relevant components change as follows:

$$
\begin{aligned}
A_y^{outer} &\;=\; A_y^{inner} - \frac{a}{Q}A_{\psi_{NS}}^{inner}, & A_\psi^{outer} &\;=\; A_{\psi_{NS}}^{inner}, \\
A_t^{outer} &\;=\; A_t^{inner} - \frac{a}{Q}A_{\phi_{NS}}^{inner}, & A_\phi^{outer} &\;=\; A_{\phi_{NS}}^{inner}.
\end{aligned}
\tag{A.2}
$$

The changes $-\frac{a}{Q}A_{\psi_{NS}}^{inner}, -\frac{a}{Q}A_{\phi_{NS}}^{inner}$ must now be compared to the terms that did not change. To do this we go to a unit orthonormal frame, where we denote components by writing a 'hat' over their index. We match the inner and outer region solutions in the throat $a \ll r \ll \sqrt{Q}$, so we use the metric (2.18). The hatted expressions are then

$$A_{\hat\psi}^{outer} \;\sim\; \frac{1}{\sqrt{Q}}A_{\psi_{NS}}^{inner}, \qquad A_{\hat y}^{outer} \;\sim\; \frac{\sqrt{Q}}{r}(A_y^{inner} - \frac{a}{Q}A_{\psi_{NS}}^{inner})\,. \tag{A.3}$$

We see that the change due to spectral flow in $A_{\hat y}^{outer}$ has magnitude

$$\Delta A_{\hat y}^{outer} \;\sim\; \frac{a}{\sqrt{Q}r}A_{\psi_{NS}}^{inner} \;\sim\; \frac{a}{r}A_{\hat\psi}^{outer}\,. \tag{A.4}$$

Assuming a general scale for all components of the perturbation $A_{\hat y}^{outer} \sim A_{\hat\psi}^{outer} \sim \hat A$, we see that

$$\frac{\Delta A_{\hat y}^{outer}}{A_{\hat y}^{outer}} \;\sim\; \frac{a}{r} \;\ll\; 1\,. \tag{A.5}$$

Thus the changes due to spectral flow are a higher order correction to the matching in the throat, and need not be considered at the leading order of matching that we perform.

B The L_{-n} perturbation in the interior

The cap geometry we have chosen is global $AdS_3 \times S^3 \times T^4$. The solution for this geometry can be scaled to take the form

$$
\begin{aligned}
ds'^2 &= -(1 + r'^2)dt'^2 + \frac{dr'^2}{1 + r'^2} + r'^2 dy'^2 + d\Omega_3^2 + dz_i' dz_i', \\
F_{abc}'^{(3)} &= 2\epsilon_{abc}, \quad F_{\mu\nu\lambda}^{(3)} = 2\epsilon_{\mu\nu\lambda},
\end{aligned} \tag{B.1}
$$

where

$$
ds'^2 = \frac{ds^2}{Q}, \quad t' = \frac{at}{Q}, \quad y' = \frac{ay}{Q}, \quad r' = \frac{r}{a}, \quad z_i' = z_i, \quad F'^{(3)} = \frac{F^{(3)}}{Q}. \tag{B.2}
$$

We will drop the primes on the variables in what follows, and restore them at the end so that we can change back to the unprimed variables.

The equations for small perturbations around $AdS_3 \times S^3$ were written down in [34]. Here we will rederive the relevant equation by assuming a simple ansatz for the perturbation. The L_{-n} perturbation is a singlet on the sphere, so we make the ansatz

$$
AdS_3 : \quad h_{\mu\nu} = M g_{\mu\nu}, \qquad S^3 : \quad h_{ab} = N g_{ab} \tag{B.3}
$$

where for now we take M, N (and P to be defined below) to be functions of t, y, r only; we will further restrict this ansatz shortly. The gauge field on the sphere remains $F_{abc}^{(3)} = 2\epsilon_{abc}$ since $\int_{S^3} F^{(3)}$ is a fixed number (the charge on the sphere). Let the field strength on the AdS be perturbed as

$$
F_{\mu\nu\lambda}^{(3)} = 2(1 + P)\epsilon_{\mu\nu\lambda}. \tag{B.4}
$$

We look for a perturbation that satisfies $F^{(3)} = *F^{(3)}$, which gives the constraint

$$
(F^{(3)})_{S^3}^2 = -(F^{(3)})_{AdS_3}^2. \tag{B.5}
$$

In the solution (B.1) we have $(F^{(3)})_{S^3}^2 = 24$. Under the perturbation above we get $(F^{(3)})_{S^3}^2 = 24(1 - 3N)$. On the AdS_3, the solution (B.1) gives $(F^{(3)})_{AdS_3}^2 = -24$. Under the perturbation this changes to $(F^{(3)})_{AdS_3}^2 = -24(1 + 2P - 3M)$. Thus the condition (B.5) gives

$$
-3N = 2P - 3M. \tag{B.6}
$$

Now we turn to the Einstein equation $R_{\mu\nu} = \frac{1}{4} F_{\mu\kappa\lambda}^{(3)} (F^{(3)})_\nu{}^{\kappa\lambda}$. For a generic perturbation h_{AB}, the change in the Ricci tensor is given by

$$
\delta R_{AB} = -\frac{1}{2}[h_{AB;C}{}^C + h_{C;AB}^C - h_{AC;B}{}^C - h_{BC;A}{}^C] \tag{B.7}
$$

so we have for the perturbation under consideration

$$
\delta R_{\mu\nu} = -\frac{1}{2}[h_{\mu\nu;\lambda}{}^\lambda + h_{A;\mu\nu}^A - h_{\mu\lambda;\nu}{}^\lambda - h_{\nu\lambda;\mu}{}^\lambda]. \tag{B.8}
$$

We have $h_A^A = 3(M + N)$. The stress tensor in the solution (B.1) is $\frac{1}{4}F_{\mu\kappa\lambda}^{(3)}(F^{(3)})_\nu{}^{\kappa\lambda} = -2g_{\mu\nu}$. Under the perturbation this changes to $-2(1 + 2P)(1 - 2M)g_{\mu\nu}$. Then the Einstein equation gives

$$-\frac{1}{2}[\Box M g_{\mu\nu} + M_{;\mu\nu} + 3N_{;\mu\nu}] \;=\; -2(2P - 2M)g_{\mu\nu}\,. \tag{B.9}$$

Only the terms $M_{;\mu\nu} + 3N_{;\mu\nu}$ are not proportional to $g_{\mu\nu}$. We thus set

$$M + 3N \;=\; 0\,. \tag{B.10}$$

Then (B.6) gives

$$P \;=\; 2M \tag{B.11}$$

and (B.9) becomes

$$(\Box - 8)M \;=\; 0\,. \tag{B.12}$$

We seek a solution to this equation of the form $M = f(r)e^{-inv}$ where $v = t - y$. The solution regular at $r = 0$ is

$$M \;=\; A\frac{r^n(n + 1 + 2r^2)}{(1 + r^2)^{\frac{n}{2}}}e^{-inv} \tag{B.13}$$

for constant A. Restoring the primes on the coordinates and then going back to the original coordinates using (B.2), this becomes

$$M \;=\; A\left(\frac{r^2}{r^2 + a^2}\right)^{\frac{n}{2}}\left(n + 1 + \frac{2r^2}{a^2}\right)e^{-in\frac{v}{R_y}}\,. \tag{B.14}$$

C   Interior solution for $J_{-n}^{(3)}$ perturbation

We now solve for the interior solution which matches onto the perturbation from the sphere diffeomorphisms. To do so we first write down the equations satisfied by the perturbation. We then rewrite these equations in terms of differential forms. Finally, we reduce the resulting equations to a single second order equation, which can be solved in closed form.

C.1   The equations satisfied by the perturbation

Following [34, 46], we make the ansatz

$$h_{a\mu}(x, y) \;=\; K_\mu(x)Y_a(y), \quad C_{\mu a}(x, y) \;=\; Z_\mu(x)Y_a(y)\,. \tag{C.1}$$

The vector spherical harmonic Y has has contravariant components

$$Y^\theta \;=\; 0, \quad Y^\psi \;=\; 1, \quad Y^\phi \;=\; -1 \tag{C.2}$$

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and satisfies

$$Y_{a;}{}^b = -2Y_a \,. \tag{C.3}$$

The relevant component of the Einstein equation is $R_{a\mu} = \frac{1}{4}F^{(3)}_{aAB}F^{(3)}{}_\mu{}^{AB}$. With the ansatz (C.1), we obtain

$$R_{a\mu} = -\frac{1}{2}[K_{\mu;\nu}{}^\nu - K_{\nu;\mu}{}^\nu - 4K_\mu]Y_a \,, \tag{C.4}$$

$$\frac{1}{4}F^{(3)}_{aAB}(F^{(3)})_\mu{}^{AB} = \frac{1}{2}[\epsilon_a{}^{bc}F^{(3)}_{\mu bc} + F^{(3)}_{a\nu\lambda}\epsilon_\mu{}^{\nu\lambda}] = \frac{1}{2}[-4Z_\mu - (Z_{\nu;\lambda} - Z_{\lambda;\nu})\epsilon_\mu{}^{\nu\lambda}]Y_a \tag{C.5}$$

where we have used

$$F^{(3)}_{\mu\nu\lambda} = 2\epsilon_{\mu\nu\lambda}, \qquad F^{(3)}_{abc} = 2\epsilon_{abc} \,. \tag{C.6}$$

Thus the Einstein equation becomes

$$[K_{\mu;\nu}{}^\nu - K_{\nu;\mu}{}^\nu - 4K_\mu] + [-4Z_\mu - (Z_{\nu;\lambda} - Z_{\lambda;\nu})\epsilon_\mu{}^{\nu\lambda}] = 0 \,. \tag{C.7}$$

The relevant component of the self-duality condition is $(*F^{(3)})_{\mu\nu a} = F^{(3)}_{\mu\nu a}$, which gives

$$\epsilon_{\mu\nu\lambda}[2K^\lambda + 2Z^\lambda] = (Z_{\nu;\mu} - Z_{\mu;\nu}) \,. \tag{C.8}$$

C.2 Rewriting the equations in 3D form language

We now denote by K, Z the 1-forms K_μ, Z_μ and we define the field strengths

$$F = dK \,, \qquad G = dZ \,. \tag{C.9}$$

In this section $*$ denotes the 3D Hodge dual. The equations (C.7), (C.8) become

$$*d*F - 4K - 4Z + 2*G = 0 \,, \tag{C.10}$$

$$2(*K + *Z) = G \,. \tag{C.11}$$

Noting that $**K = -K$ and similarly for Z, we can write (C.11) as $2(K+Z) = -*G$. Substituting this in (C.10) we get

$$*d*F + 4*G = 0 \quad \Rightarrow \quad d*F + 4\,dZ = 0 \,. \tag{C.12}$$

This has the solution $Z = -\frac{1}{4}*F$ up to gauge transformations $Z \to Z + d\Lambda_Z$. We write this as

$$F = 4*Z \,. \tag{C.13}$$

Then from (C.11) we get

$$K = -Z - \frac{1}{2}*G \,. \tag{C.14}$$

Applying the operator d to both sides of this relation gives $F = -G - \frac{1}{2}d * G$. Using (C.13) we get a relation involving Z alone: $4 * Z = -G - \frac{1}{2}d * G$. In terms of Z, this is $*d * dZ + 2 * dZ - 8Z = 0$, which can be written as

$$(*d - 2)(*d + 4)Z = 0. \tag{C.15}$$

This splits into a pair of equations

$$(*d - 2)Z = 0, \quad \text{or} \quad (*d + 4)Z = 0. \tag{C.16}$$

Below we will solve the general equation

$$* dZ + \alpha Z = 0. \tag{C.17}$$

We will see that the solution we need corresponds to the choice $\alpha = -2$. For this value of α we note from (C.14) that

$$K = -2Z. \tag{C.18}$$

C.3 Solving the equations

We wish to solve the equation (C.17) with an ansatz of the form

$$Z_v = f(r)e^{-inv}, \qquad Z_u = q(r)e^{-inv}, \qquad Z_r = h(r)e^{-inv}. \tag{C.19}$$

Then the field strength $G = dZ$ has nonzero components

$$G_{rv} = (f' + inh)e^{-inv}, \qquad G_{ru} = q'e^{-inv}, \qquad G_{uv} = inqe^{-inv}. \tag{C.20}$$

The v, u and r components of the equation $*G = -\alpha G$ then give the equations

$$\frac{1}{2r}\left[-q' + (2r^2 + 1)(f' + inh)\right] = -\alpha f, \tag{C.21}$$

$$-\frac{1}{2r}\left[(2r^2 + 1)q' - (f' + inh)\right] = -\alpha q, \tag{C.22}$$

$$\frac{2in}{r(r^2 + 1)}q = -\alpha h. \tag{C.23}$$

The last relation (C.23) can be immediately solved to give

$$h = -\frac{2in}{\alpha r(r^2 + 1)}q. \tag{C.24}$$

Rearranging (C.21) we get

$$f' + inh = \frac{-2r\alpha f + q'}{2r^2 + 1} \tag{C.25}$$

which we substitute in (C.22) to get

$$f = -\frac{2r(r^2 + 1)}{\alpha}q' + (2r^2 + 1)q. \tag{C.26}$$

Substituting f back in (C.21) we get

$$r^2(r^2 + 1)^2 q'' + r(r^2 + 1)(3r^2 + 1)q' - \left[\alpha(\alpha + 2)r^2(r^2 + 1) + n^2\right]q = 0. \tag{C.27}$$

C.4 Selecting a solution

We now want to solve (C.27). The equation for the perturbation (C.16) allowed the values $\alpha = -2, \alpha = 4$. The solution which matches on to the perturbation we construct in the neck is the $\alpha = -2$ solution, which has the form

$$q = C\left(\frac{r^2}{1+r^2}\right)^{\frac{n}{2}} + D\left(\frac{1+r^2}{r^2}\right)^{\frac{n}{2}}. \tag{C.28}$$

We see that regularity at the origin requires n to be an integer. For $n > 0$, we set $D = 0$, getting $q = C(\frac{r^2}{1+r^2})^{\frac{n}{2}}$.

So the solution is

$$\begin{aligned}
Z_v &= C\left(\frac{r^2}{r^2+1}\right)^{\frac{n}{2}}\left(n+1+2r^2\right)e^{-inv}, \\
Z_u &= C\left(\frac{r^2}{r^2+1}\right)^{\frac{n}{2}}e^{-inv}, \\
Z_r &= C\frac{in}{r(r^2+1)}\left(\frac{r^2}{r^2+1}\right)^{\frac{n}{2}}e^{-inv}.
\end{aligned} \tag{C.29}$$

Restoring the primes on the coordinates and then going back to the original coordinates using (B.2), we get

$$\begin{aligned}
Z_v &= C\,a\left(\frac{r^2}{r^2+a^2}\right)^{\frac{n}{2}}\left(n+1+\frac{2r^2}{a^2}\right)e^{-in\frac{v}{R_y}}, \\
Z_u &= C\,a\left(\frac{r^2}{r^2+a^2}\right)^{\frac{n}{2}}e^{-in\frac{v}{R_y}}, \\
Z_r &= C\,in\frac{a^2Q}{r(r^2+a^2)}\left(\frac{r^2}{r^2+a^2}\right)^{\frac{n}{2}}e^{-in\frac{v}{R_y}},
\end{aligned} \tag{C.30}$$

and from (C.18) we recall that for $\alpha = -2$ we have

$$K_\mu = -2Z_\mu. \tag{C.31}$$

D Interior solution for J^i_{-n} perturbation

We now solve for the interior solution which matches onto the perturbation from the torus diffeomorphism. We make the ansatz

$$h_{\mu z} = K_\mu, \quad C_{\mu z} = Z_\mu. \tag{D.1}$$

We define the field strengths

$$F_{\mu\nu} = \partial_\mu K_\nu - \partial_\nu K_\mu, \quad G_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \tag{D.2}$$

The relevant component of the Einstein equation is $R_{z\mu} = \frac{1}{4}F^{(3)}_{zAB}F^{(3)AB}_{\mu}$ which gives

$$F_{\mu\nu}{}^{;\nu} = \frac{1}{2}G_{\mu_1\mu_2}(\bar{F}^{(3)})^{\mu_1\mu_2}{}_{\mu} \tag{D.3}$$

where $\bar{F}^{(3)}$ is the background value of $F^{(3)}$. The field equation $F^{(3)}_{z\mu A}{}^{;A} = 0$ gives

$$G_{\mu\nu}{}^{;\nu} - \frac{1}{2}F_{\mu_1\mu_2}\bar{F}^{(3)\mu_1\mu_2}{}_{\mu} = 0. \tag{D.4}$$

Again using $*$ for the 3D Hodge dual, the Einstein equation in terms of forms is

$$* \, d * F = -2 * G \tag{D.5}$$

and the field equation is

$$* \, d * G = -2 * F. \tag{D.6}$$

Adding and subtracting these equations, we get two decoupled equations

$$\begin{array}{llll}
* \, d * (F + G) = & -2 * (F + G) & \Rightarrow & d * (F + G) = & -2(F + G) \\
* \, d * (F - G) = & 2 * (F - G) & \Rightarrow & d * (F - G) = & 2(F - G)
\end{array} \tag{D.7}$$

These equations are of the form (C.17), i.e. of the form $d * H + \alpha H = 0$ for a 2-form H. If $H = dA$, then we get $d * dA + \alpha dA = 0$. This gives

$$* \, dA + \alpha A = 0 \tag{D.8}$$

up to gauge transformations $A \to A = d\Lambda_A$. Thus the 1-forms $K + Z, K - Z$ satisfy (C.17) with $\alpha = 2, -2$ respectively.

We find that the solution which will matches to the solution in the outer region is obtained by choosing $K + Z = 0$. Then the equation for $K - Z = -2Z$ is the same as the equation for Z solved in section C.4, namely $(*d - 2)Z = 0$, so the solution for Z has the same form as for the sphere case:

$$\begin{aligned}
Z_v &= D\,a\left(\frac{r^2}{r^2 + a^2}\right)^{\frac{n}{2}}\left(n + 1 + \frac{2r^2}{a^2}\right)e^{-in\frac{v}{R_y}}, \\
Z_u &= D\,a\left(\frac{r^2}{r^2 + a^2}\right)^{\frac{n}{2}}e^{-in\frac{v}{R_y}}, \\
Z_r &= D\,in\frac{a^2Q}{r(r^2 + a^2)}\left(\frac{r^2}{r^2 + a^2}\right)^{\frac{n}{2}}e^{-in\frac{v}{R_y}}
\end{aligned} \tag{D.9}$$

for some constant D. This solution has $K + Z = 0$, so K is given by

$$K_\mu = -Z_\mu. \tag{D.10}$$

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